NUMBER THEORY AND DIOPHANTINE ANALYSIS.

150. Proposed by H. S. VANDIVER, Bala, Pa.

Show that for all positive integral values of n except unity, $(2n)!$ is less than $[n(n+1)]^n$. Direct proof preferred. [Unsolved problem in *Educational Times.*]

I. Solution by JACOB WESTLUND, Purdue University, Lafayette, Ind., and the PROPOSER.

If $0<k<n$, we have $(n-k)(n+k+1)<n(n+1)$, since $n^2+n-(k^2+k)<n^2+n$.

Hence, letting k run through the values 0, 1, 2, ..., $n-1$, we get $(2n)!<[n(n+1)]^n$.

II. Solution by W. F. KING, Ottawa, Canada.

$$(2n)!=(1.2.3...n)[(n+1)(n+2)...(2n)]$$
$$=[n(n-1)(n-2)...(n-\overline{n-1})]$$
$$\times[\overline{n+1}(\overline{n+1}+1)(\overline{n+1}+2)...(\overline{n+1}+\overline{n-1})].$$
$$\therefore\frac{(2n)!}{[n(n+1)]^n}=\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-1}{n}\right)\right]$$
$$\times\left[\left(1+\frac{1}{n+1}\right)\left(1+\frac{2}{n+1}\right)...\left(1+\frac{n-1}{n+1}\right)\right].$$

Taking any factor $1-\dfrac{r}{n}$ in the first bracket with the corresponding factor $1+\dfrac{r}{n+1}$ in the second bracket, their product is $\left(1-\dfrac{r}{n}\right)\left(1+\dfrac{r}{n+1}\right)$ $=1-\dfrac{r}{n(n+1)}-\dfrac{r^2}{n(n+1)}$, which is <1.

The product of each pair of terms being <1, the whole product is <1.

$$\therefore\frac{(2n)!}{[n\ (n+1)]^n}<1,\text{ and }(2n)!<[n(n+1)]^n.\quad\text{Q. E. D.}$$

III. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Stating the problem in the form $\dfrac{[n(n+1)]^n}{(2n)!}>1$, it is evidently true for $n=2$, *i. e.*, $\frac{3}{2}>1$. The factor which will change the first member of the first inequality into the form in which n becomes $n+1$ is

$$F=\frac{(n+2)^{n+1}}{2\ (2n+1)n^n}=\left(\frac{n+2}{n}\right)^n\cdot\frac{n+2}{2\ (2n+1)}=\left[\left(1+\frac{2}{n}\right)^{\frac{1}{2}n}\right]^2\cdot\frac{(1+2/n)}{(4+2/n)}.$$

Since $\left(1+\frac{2}{n}\right)^{\frac{1}{2}n}>2$, and $\frac{1+2/n}{4+2/n}>\frac{1}{4}$, it follows that $F>1$, hence for successive values of n the original inequality becomes stronger.

It is of interest to notice that the limiting value of F, for $n=\infty$, is $\varepsilon^2/4$, but this is not essential to the proof.

Excellent solutions of this problem were received from G. B. M. Zerr, Frank L. Griffin, C. E. White, and O. L. Callicot.

151. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the recurring series, $n=0, 1, 2, 3, 4, 5, 6, 7, \ldots$
$u_n=3, 0, 2, 3, 2, 5, 5, 7, \ldots$
where the scale of relation is $u_{n+3}=u_{n+1}+u_n$, prove that u_p is always divisible by p when p is prime. Is the converse true?

Solution by the PROPOSER.

The general term of the series is $u_n=\alpha^n+\beta^n+\gamma^n$ where α, β, γ are roots of the equation $x^3-x-1=0$.

When p is prime, $(\alpha+\beta+\gamma)^p\equiv\alpha^p+\beta^p+\gamma^p$ (mod. p), and since $\alpha+\beta+\gamma=0$, we have $u_p\equiv0$ (mod. p).

I believe the converse is not true, although I have not found an example which disproves it.

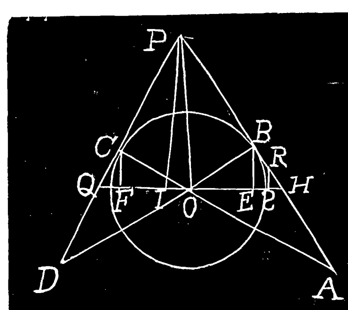
MECHANICS.

213. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two unequal, uniform, smoothly hinged rods are placed over a smooth vertical circle. Apply the principle of virtual work to find the condition of equilibrium in terms of the length of each rod, the diameter of the circle and the angle of either rod with the vertical.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let O be the center of the vertical circle; PD, PA, the rods jointed at P; C, B, the points of tangency of PD, PA with the circle; Q, H the intersection of the horizontal diameter with PD, PA. Let r=radius of circle, $PD=l$, $PA=m$. Draw PI, CF, BE perpendicular to HQ. Let $\angle BOE=\angle HBE=\phi$, $\angle COF=\angle FCQ=\angle IPQ=\theta$. Then $PC=PB=r\cot\frac{1}{2}(\phi+\theta)$, $CQ=r\tan\theta$, $BH=r\tan\phi$, $OI=\dfrac{r\sin\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}(\phi+\theta)}$.



Let z=the height of the center of gravity of the system above O. Also for equilibrium the resultant of the weights of the rods must pass through O.